ST Yau College Students Mathematics Contest Applied and Computational Math (Individual Final)

June 10, 2023

1. Let $\boldsymbol{U} \in \mathbb{R}^{n\times n}$ be an orthogonal matrix satisfying $\det(\boldsymbol{U}) = 1$.

   (a) Prove that $\boldsymbol{U}$ can be written into the product of finitely many Givens rotation matrices. Recall that an $n \times n$ Givens rotation matrix is an orthogonal matrix $\boldsymbol{G}(i, j, \theta)$, for some given indices $i > j$ and some angle $\theta \in [0, 2\pi]$, whose entries are the same as the identity matrix except for

   $$\begin{cases} g_{ii} = g_{jj} = \cos\theta, \\ g_{ij} = -g_{ji} = \sin\theta. \end{cases}$$

   (b) Find an algorithm to compute the Givens decomposition in part (a).

2. Let $A \in \mathbb{C}^{n\times n}$ be a self-adjoint matrix wth k dominant eigenvalues, which are denoted by λ_j, $j = 1, 2, \cdots, n$. In particular, we have

   $$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_k| > |\lambda_{k+1}| \geq \cdots \geq |\lambda_n|.$$

   We write

   $$A = QDQ^*$$

   where $Q \in \mathbb{C}^{n\times n}$ is unitary and $D = \mathrm{diag}(\lambda_j) \in \mathbb{C}^{n\times n}$ is diagonal. Consider the following iteration

   $$X^{(m+1)} = AX^{(m)}.$$

   Assume that $X^{(0)} \in \mathbb{C}^{n\times k}$ is given. Define $\widehat{P} \in \mathbb{C}^{n\times n}$ by

   $$\widehat{P} = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$$

   where I_k is the $k \times k$ identity matrix, and $P = Q\widehat{P}Q^*$. Assume that $PX^{(0)}$ has independent columns.

(a) Show that $PX^{(m)}$ also has independent columns.

(b) Hence, show that $X^{(m)}$ has independent columns.

(c) Show that, there is a matrix $\Lambda \in \mathbb{C}^{k\times k}$ such that

$$\frac{\|(AX^{(m)} - X^{(m)}\Lambda)y\|}{\|PX^{(m)}y\|} \leq \left(\frac{|\lambda_{k+1}|}{|\lambda_k|}\right)^m \frac{\|(AX^{(0)} - X^{(0)}\Lambda)y\|}{\|PX^{(0)}y\|}$$

for all non-zero $y \in \mathbb{C}^k$.

3. Consider a system of two ODEs of the form

$$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y).$$

Suppose that it is more computationally expensive to evaluate g than to evaluate f.

(a) Prove that the multi-rate explicit Euler method defined by

$$\begin{aligned} x_{j+1/2} &= x_j + \frac{k}{2} f(x_j, y_j), \\ x_{j+1} &= x_{j+1/2} + \frac{k}{2} f(x_{j+1/2}, y_j), \\ y_{j+1} &= y_j + kg(x_j, y_j), \end{aligned}$$

is locally second order, where k is the time step.

(b) Consider applying the method from (a) to the following linear problem:

$$\frac{dx}{dt} = -x + y, \quad \frac{dy}{dt} = -y.$$

Under what conditions on the time step k will the discrete solution remain stable, i.e., as $j \to \infty$, both $x_j \to 0$ and $y_j \to 0$ for any initial conditions?

4. For the advection equation $u_t + au_x = 0$ with $a > 0$, consider the five-point stencil:

$$u_j^{n+1} = \frac{1}{2}(u_{j-1}^n + u_{j+1}^n) + \frac{ak}{12h}(u_{j+2}^n - 8u_{j+1}^n + 8u_{j-1}^n - u_{j-2}^n).$$

(a) Recall that the CFL condition for a scheme is when the numerical domain of dependence contains the analytic domain of dependence. It is a necessary condition for stability but not sufficient. Write down the CFL condition for this scheme.

(b) Write down its amplification factor $g(\omega)$. Recall that the von Neumann stability Condition for a scheme is the condition on a, k, h such that $|g(\omega)| < 1 + Kk$ for all admissible ω, h, k. It is a necessary and sufficient condition for stability. In this case, we see $|g(\omega)|$ depends on k through $\lambda = k/h$, so the criterion reduced to $|g(\omega)| \leq 1$ for all admissible ω, k, h. Find out the von Neumann condition for this scheme. How does that compare with the CFL condition?